2020

MATHEMATICS

[HONOURS]

Paper : VI

Full Marks : 100 Time : 4 Hours

The figures in the right-hand margin indicate marks. Candidates are required to give their answers in their own words as far as practicable.

Symbols and notations have their usual meanings.

1. Answer any **five** questions: $1\times5=5$

a) Show that the function [x] where [x] denotes the greatest integer not greater than x is integrable in [0, 3] and also $\int_0^3 [x]dx = 3$.

b) Find the interval of absolute convergence for the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^n}.$$

c) Prove that the series $\sum_n (-1)^n \left(\frac{x^2+n}{n^2}\right)$ converges uniformly in every bounded interval but does not converge absolutely for any value of x.

d) Prove that the function $f(x,y)=\sqrt{|xy|}$ is not differentiable at the point (0, 0), but that f_x and f_y both exist at the origin.

e) Examine the convergence of $\int_1^{\infty} \frac{dx}{x\sqrt{x^2+1}}$.

f) Eliminate the arbitrary function ϕ from $z = e^{my}\phi(x-y)$.

g) Show that

$$\lim_{n\to\infty}\int_0^a \phi\left(\frac{\sin nx}{\sin x}\right)dx = \lim_{n\to\infty}\int_0^a \phi\left(\frac{\sin nx}{x}\right)dx.$$

2. Answer any **ten** questions: $2\times10=20$

a) If a function f is continuous on [0, 1] then show that

$$\lim_{n\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x^2}dx = \frac{\pi}{2}f(0).$$

b) Prove that the integral $\int_0^{\frac{\pi}{2}} \left(\frac{\sin^m x}{x^n}\right)dx$ exists if and only if $n < m+1$.

c) Show that $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$ is invariant for change of rectangular axes.

d) The function x^2 is periodic with period $2l$ on the interval [–1 l]. Find its Fourier series.

e) If f is continuous on [0, 1] and if $\int_0^1 x^n f(x)dx = 0$ for n = 0,1,2,... then show that f(x)=0 on [0, 1].

f) Show that the function $f(x,y) = 2x^4 - 3x^2y + y^2$ has neither a maximum nor a minimum at (0, 0).

g) Prove that x = 0 is an ordinary point of $(x^2-1)y'' + xy' - y = 0$, but x=1 is a regular singular point.

h) Evaluate the integral

$$I = \int_0^1 dx \int_0^x \sqrt{x^2+y^2}\, dy$$

by passing on to the polar coordinates.

i) Find the partial differential equation of all surfaces of revolution, having z-axis as the axis of revolution.

j) Find the value of $\int_c \left\{(x+y^2)dx + (x^2-y)dy\right\}$ taken in the clockwise sense along the closed curve C formed by $y^3 = x^2$ and the chord joining (0, 0) and (1, 1).

k) If f is bounded and integrable on [a, b] then show that $\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$.

l) Find the Laplace transform of the function $F(t) = (t^2 - 3t + 2)\sin 3t$.

3. Answer any **five** questions: 6×5=30

a) Test for the convergence of the integral

$$\int_0^1 x^p \left(\log\frac{1}{x}\right)^q dx.$$

b) Expand the periodic function. of period $2l$>0,

$f(x) = \left|\cos\left(\frac{\pi x}{l}\right)\right|$, in a Fourier series.

c) Prove that, by the transformations $u = x - ct,\ v = x + ct,$ the partial differential equation $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial t^2}$ reduces to $\frac{\partial^2 z}{\partial u \partial v} = 0$.

d) Show that the volume of the greatest rectangular parallelopiped, that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{8abc}{3\sqrt{3}}$.

e) Find the power series solution of the equation $(x^2+1)y''+xy'-xy=0$ in powers of x (i.e., about x=0)

f) i) Show that the differential equation of all cones which have their vertex at the origin is $px+qy=z$. Verify that $yz+zx+xy=0$ is a surface satisfying the above equation.

ii) Solve :

$$(3x+y-z)p+(x+y-z)q=2(z-y)$$

where $p\equiv\frac{\partial z}{\partial x}$ and $q\equiv\frac{\partial z}{\partial y}$. 3+3

g) i) Prove that the set C[0, 1] consisting of all real-valued continuous functions defined on [0, 1] with the function d given by

$$d(f,g)=\int_0^1|f(x)-g(x)|dx \;\forall f,g\in c[0,1]$$

is a metric space.

ii) Show by means of an example, the arbitrary union of closed sets in a metric space is not necessarily a closed set.

4+2

h) i) If z = f(x,y) possesses n^{th} order partial derivatives and x, y are linear functions of a single variable t i.e. x = a+ht, y=b+kt where a, b, h, k are constants then prove that $\frac{d^n z}{dt^n}=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n z$.

ii) If $f(x,y)=\sqrt{|xy|}$, prove that Taylor's expansion about the point (x,x) is not valid in any domain which includes the origin. 3+3

4. Answer any **three** questions: 15×3=45

a) i) Prove that a bounded function f, having a finite number of points of discontinuity on [a, b] is integrable on [a, b]

ii) A function f is defined on [0, 1] as follows:

$$f(x)=\begin{cases}0,\text{ when x is irrational or zero, and}\\ 1/v,\text{ when x is any non}-\text{zero rational number } p/v \text{ with}\\ \text{least positive integers p and v.}\end{cases}$$

show that f is integrable on [0, 1] and the value of the integral is zero.

iii) If f is monotone and f, f' and g are all continuous in [a, b] then prove that there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^{\xi} g(x)dx + f(b)\int_{\xi}^b g(x)dx.$$

5+5+5

b) i) Given that F is a function of x and y and that $x = e^u + e^{-v}$, $y = e^v + e^{-u}$, then prove that

$$\frac{\partial^2 F}{\partial u^2} - 2\frac{\partial^2 F}{\partial u \partial v} + \frac{\partial^2 F}{\partial v^2} = x^2\frac{\partial^2 F}{\partial x^2} - 2xy\frac{\partial^2 F}{\partial x \partial y} + y^2\frac{\partial^2 F}{\partial y^2} + x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y}.$$

ii) If $u = \frac{x^2 + y^2 + z^2}{x}, v = \frac{x^2 + y^2 + z^2}{y}$ and $w = \frac{x^2 + y^2 + z^2}{z}$ then find

$$\frac{\partial(x, y, z)}{\partial(u, v, w)}.$$

iii) If the variables x, y,z satisfy the equation $\phi(x)\phi(y)\phi(z) = k^3$ and $\phi(a) = k \neq 0$, $\phi'(a) \neq 0$, then show that the function $f(x) + f(y) + f(z)$ has a minimum when $x = y = z = a$, provided that

$$f'(a)\left\{\frac{\varphi''(a)}{\varphi'(a)} - \frac{\varphi'(a)}{\varphi(a)}\right\} > f''(a).$$

5+4+6

c) i) Solve :

$$(x^2 + y^2 + yz)p + (x^2 + y^2 - xz)q = z(x + y)$$

where $p \equiv \frac{\partial z}{\partial x}$ and $q \equiv \frac{\partial z}{\partial y}$.

ii) Find a complete and singular integrals of $2xz - px^2 - 2qxy + pq = 0$.

iii) Use Laplace transforms to solve the following problem:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = e^{-3t},\ y(0) = y'(0) = 0.$$

5+5+5

d) i) Find the power series solution of the initial value problem

$$(x^2 - 1)y'' + 3xy' + xy = 0,\ y(2) = 4,\ y'(2) = 6.$$

ii) Evaluate

$$\iint_E x^{m-1}y^{n-1}(1 - x - y)^{p-1}dxdy,\ m \geq 1,\ n \geq 1,\ p \geq 1$$

where E is the region bounded by $x = 0,\ y = 0,\ x + y = 1$.

iii) If a series Σf_n converges uniformly to f in an interval [a, b] and its terms f_n are continuous at a point x_0 of the interval then show that the sum function f is also continuous at x_0. 5+6+4

e) i) Prove that every closed subset of a compact metric space is compact.

ii) Let l_∞ be the set of all bounded numerical sequences $\{x_n\}$ in which the metric d is defined by

$$d(x, y) = \sup_n |x_n - y_n|, \forall x = \{x_n\}, y = \{y_n\} \in l_\infty .$$

Then show that (l_∞, d) is a complete metric space.

iii) Prove that

$$\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3} dy = \frac{1}{2}, \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} dx = -\frac{1}{2}.$$

Does the double integral $\iint_R \frac{x-y}{(x+y)^3} dxdy$ exist over R=[0, 1; 0, 1]

4+6+4+1

f) i) Find the expansion of sin x sin y about (0, 0) up to and including the terms of the fourth degree in (x, y). Compare the result with that you get by multiplying the series for sin x and sin y.

ii) Show that if f, f_x, f_y are all continuous in a domain D of (a, b) and D is large enough to contain the point (a+h, b+k) within it, then for $0 < \theta < 1$,

$$f(a+h, b+k) = f(a, b) + hf_x(a+\theta h, b+\theta h) + kf_y(a+\theta h, b+\theta h).$$

If $f(x, y) = x\sqrt{x^2 + y^2}$, $a = b = -1, h = k = 3$, verify that the above conditions are satisfied and find the value of θ.

iii) Test the convergence of $\int_0^\infty \frac{\sin x^m}{x^n} dx$.

(4+2)+(4+2)+3

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